Category-Theoretic Radical Ontic Structural Realism

Jonathan Bain

Department of Technology, Culture and Society
Tandon School of Engineering, New York University
Brooklyn, New York

1. No Structures Without Objects?
2. An Analogy From General Relativity.
3. How To Do Category-Theoretic Physics.

1. No Structures Without Objects?

Radical Ontic Structural Realism, 1.0 (French & Ladyman 2003)

Physical structure consists of relations devoid of *relata*.

Subclaim 1: Physical structures are physical relations.

Subclaim 2: Physical relations can exist without physical *relata*.

1. No Structures Without Objects?

Radical Ontic Structural Realism, 1.0 (French & Ladyman 2003)
Physical structure consists of relations devoid of *relata*.

"...when it comes to the physical world, the point at issue are concrete relations that are instantiated in the physical world and that hence are particulars in contrast to universals. For the relations to be instantiated, there has to be something that instantiates them... ." (Esfeld & Lam 2008)

"As applied to a particular relation, this assertion seems incoherent. It only makes sense if it is interpreted as the metaphysical claim that ultimately there are only relations; that is, in any given relation, all of its *relata* can in turn be interpreted as *relations*." (Stachel 2006)

"I daresay that no ontic structural realist should be falling into the trap of accepting the view that 'relations can exist without *relata*'." (Dorato 2008)

1. No Structures Without Objects?

Radical Ontic Structural Realism, 2.0 (French 2014)
Physical structure exists independently of physical objects.

"...one crucial point is that, even if structures are to be both abstract (laws and symmetries) and concrete, one can still with good reason expect OSR to be able to draw a distinction between (abstract) mathematical and (concrete) physical structure." (Esfeld 2015)

"...the specter of Pythagoreanism rears its head. If the structure of the world is all there is, and this structure is group-theoretic [say], then how is this not to say that the world is fundamentally mathematical?" (Ney 2014)

"...my impression is that the main objective has not been reached. The reason has to do with the fact that the simple question 'what is *physical* rather than mere *mathematical* structure?' has received at best a vague answer." (Dorato 2016)

1. No Structures Without Objects?

The Mathematical Question:
Are there mathematical (abstract) representations of structures that are independent of objects?

The Physical Question:
Why should we think there are physical (concrete) structures that are independent of objects?

1. No Structures Without Objects?

Claim 1 (Mathematical): Category theory provides the means of mathematically representing structures independently of objects.

Claim 2 (Physical): *One* reason to believe that there are physical structures that are independent of objects is that there are theories in physics that

(a) warrant our belief,

(b) we should read literally, and

(c) employ category theory to represent structures independently of objects.

1. No Structures Without Objects?

Radical Ontic Structural Realism, 2.0

Physical structure exists independently of physical objects.

Untenable?

If metaphysical intuitions are informed by *set-theory*, then perhaps so.

- Let a *physical structure* be represented by a set-theoretic structure = isomorphism class of structured sets = $[\{X, R_i\}]$.
- Let a *physical object* be represented by an element of X.

  - A (*binary*) *relation* R on X is a subset of $X \times X$, the set of all ordered pairs (x, y), $x, y \in X$.
  - An *ordered pair* (x, y) is the set $\{x, \{x, y\}\}$.
  - Ineliminable reference to elements of a set.

1. No Structures Without Objects?

Radical Ontic Structural Realism, 2.0
Physical structure exists independently of physical objects.

Untenable?

If metaphysical intuitions are informed by *category theory*, then perhaps not.

- *Primitives:* C-objects, morphisms between C-objects.
- Reference to *internal* constituents of a C-object can only be done in terms of *external* C-objects and morphisms.

"In category theory, it is by conditions on the web of morphisms of a category that internal structure is imposed on the [C-]objects of the category." (Lam & Wüthrich 2015)

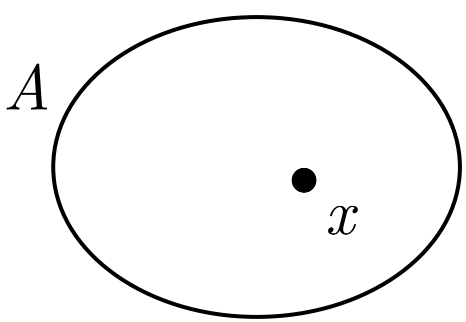
1. No Structures Without Objects?

A *C-element* x of a C-object A in a category **C** is a morphism $\mathbf{1} \rightarrow A$, where $\mathbf{1}$ is a *terminal C-object* in **C**.

A generalized *C-element* x of a C-object A in **C** is a morphism $U \rightarrow A$, where U is some standard C-object in **C**.

Set Theory

Primitives: sets, $\in$



$x \in A$

Category Theory

Primitives: C-objects, morphisms

$$\mathbf{1} \xrightarrow{x} A$$

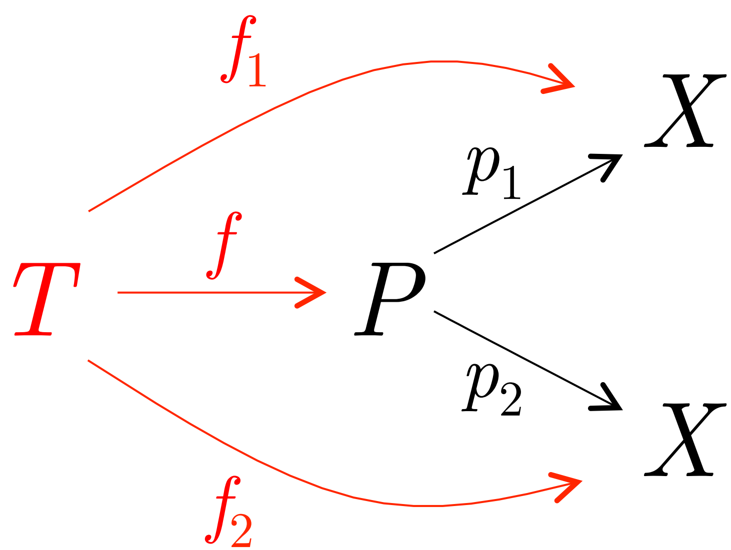
For A a C-object in **Set**

$$\mathbb{Z} \xrightarrow{x} A$$

For A a C-object in **Grp**

1. No Structures Without Objects?

The *Cartesian product* of a C-object X with itself is a C-object P, together with a pair of morphisms $p_1 : P \to X$, $p_2 : P \to X$ such that, for any C-object T with morphisms $f_1 : T \to X$, $f_2 : T \to X$, there is exactly one morphism $f : T \to P$ for which $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$.



- External probe (T, f_1, f_2, f) encodes internal structure of P.

1. No Structures Without Objects?

Radical Ontic Structural Realism, 2.0

Physical structure exists independently of physical objects.

- Let a *physical structure* be represented by the "web of morphisms" of a category **C**.
- Let a *physical object* be represented by a (generalized) C-element of a C-object in a category **C**.

Then: Physical structures can exist independently of physical objects to the extent that

(a) there are categories in which (generalized) C-elements do not occur; and,

(b) there are categories in which (generalized) C-elements occur, but do not figure into the essential web of morphisms that defines the structure of the category.

1. No Structures Without Objects?

Objection 1: "...given that categories are defined in terms of arrows (morphisms) *and objects*, category theory is not a framework that an OSR-theorist can adopt to answer the question regarding the nature of structures. Similarly to set theory, it is ultimately an object-oriented view." (Arenhart & Bueno 2015)

Response #1

- Suppose "object-oriented view" refers to "physical object-oriented view".
- To the extent that the primitives of category theory do not include (generalized) C-elements of C-objects, category theory need not be a physical object-oriented view.

1. No Structures Without Objects?

Objection 1: "...given that categories are defined in terms of arrows (morphisms) *and objects*, category theory is not a framework that an OSR-theorist can adopt to answer the question regarding the nature of structures. Similarly to set theory, it is ultimately an object-oriented view." (Arenhart & Bueno 2015)

Response #2

- Suppose "object-oriented view" refers to "C-object-oriented view".
- Category theory can be formulated solely in terms of morphisms, with no reference to C-objects. (Mac Lane 1998)

1. No Structures Without Objects?

Objection 2: Elimination of objects in name only.

- Where set theory sees "elements", category theory sees "morphisms from a terminal/standard C-object".
- "*No structures without objects*" becomes
  - *Set-theory*: *"No isomorphism class of structured sets without elements".*
  - *Category-theory*: *"No web of morphisms without morphisms from the terminal/standard C-object".*

Response

- There are categories in which the C-objects are not structured sets.
- In such categories, (generalized) C-elements may or may not occur, *but regardless, they are not an essential part of the "web of morphisms" that defines the category.*

2. An Analogy from General Relativity

Tensor formalism		**Einstein algebra formalism**
$(M,\ g_{ab})$	$\overset{\text{1-1}}{\longleftrightarrow}$	$(\mathcal{C},\ g)$

differentiable manifold

metric field satisfying Einstein equations

points of M correspond to maximal ideals of $\mathcal{C}$

commutative ring

multilinear map on space of derivations of $\mathcal{C}$ and its dual, satisfying Einstein equations

- *Idea:* Reconstruct M as collection of maximal ideals of commutative ring $C^\infty(M)$ of smooth functions on M.
- *Different object-based ontologies:* Points *vs.* ideals.
- *Common structure:* Differentiable structure.

Elimination of points in name only?

2. An Analogy from General Relativity

Consider: GR with asymptotic boundary conditions.

Tensor Models

- Replace M with *manifold with boundary* $M' = M \bigcup \partial M$.
- $(M,\ g_{ab})$ is Diff(M)-invariant.
- $(M',\ g_{ab})$ is Diff$_c(M)$-inv., but *not necessarily* Diff(M)-inv.

  diffeomorphisms on M with compact support $\approx$ *"local" diffeomorphisms*

- No morphisms that preserve *both* M and M'.
- M and M' belong to *different* categories.

2. An Analogy from General Relativity

Consider: GR with asymptotic boundary conditions.

Einstein Algebra Models

- Replace *ring* $\mathcal{C} \cong C^\infty(M)$ with *sheaf* $\mathscr{C} \cong C^\infty(M')$.
- Replace *Einstein algebra* $(\mathcal{C}, g)$ with *sheaf of Einstein algebras* $(\mathscr{C}, g)$.
  - $(\mathscr{C}, g)$ does not in general have *global cross sections* (*i.e.*, C-elements).
  - $(\mathcal{C}, g)$ and $(\mathscr{C}, g)$ are C-objects in the *same* category: the category of "Einstein structured spaces". (Heller & Sasin 1995)

2. An Analogy from General Relativity

Upshot:

- Structure of tensor models: *"local" differentiable structure.*
  - Predicated directly on points of M.

- Structure of EA models: *"global" differentiable structure.*
  - Encoded directly in a sheaf of Einstein algebras.
  - *Not* predicated on maximal ideals of a single Einstein algebra.

- Tensor models (M, g_{ab}) are structured sets.
- Einstein structured spaces $(\mathscr{C}, g)$ are not!

2. An Analogy from General Relativity

Thus:

(1) The point correlates (maximal ideals) in Einstein algebra models of GR do not play an essential role in articulating the relevant notion of structure.

(2) Einstein algebra models of GR provide a more unifying description of phenomena in GR.

Analogously:

(1') The correlates of set-theoretic elements in category theory do not play an essential role in articulating the relevant (category-theoretic) notion of structure.

(2') This notion of structure does actual work in providing a more unifying description of phenomena.

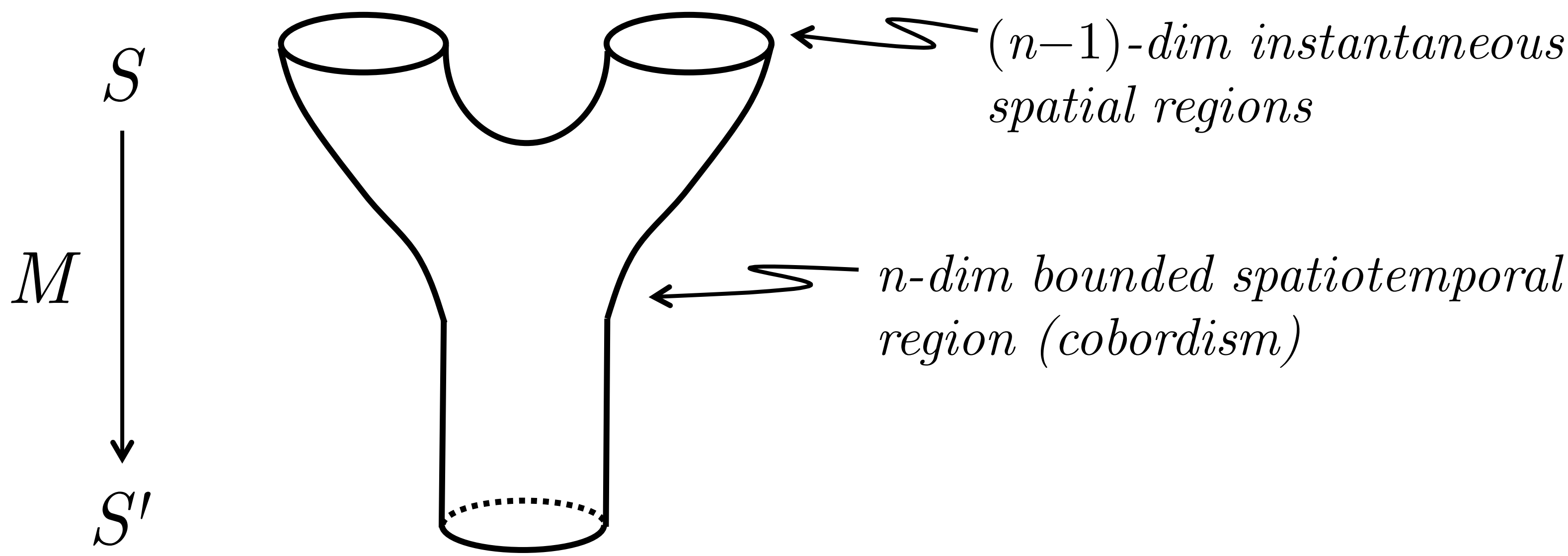
3. How To Do Category-Theoretic Physics

- Two more examples of categories, n**Cob** and **Hilb**. (Baez 2006)
  - The C-elements of the C-objects in these categories, while well-defined, do not play an essential role in articulating the relevant category-theoretic notions of structure.
  - *Moreover:* These category-theoretic notions of structure are relevant to the pursuit of unifying descriptions of physical phenomena.

3. How To Do Category-Theoretic Physics

Ex. 1: The category ***n*Cob**.

- C-objects: $(n-1)$-dim topological manifolds.
- Morphisms: n-dim topological manifolds with boundary.



3. How To Do Category-Theoretic Physics

Ex. 1: The category $\boldsymbol{n}$**Cob**.

- C-objects: $(n-1)$-dim topological manifolds.
- Morphisms: n-dim topological manifolds with boundary.

Set-theoretically:

- Topological spaces are structured sets.
- Structure-preserving functions are homeomorphisms.

Category-theoretically:

- C-objects of $\boldsymbol{n}$**Cob** are not structured sets: morphisms are not even functions.
- Unlike **Set**, $\boldsymbol{n}$**Cob** admits a tensor product but no Cartesian product.
- C-elements of $\boldsymbol{n}$**Cob** are manifold points, but manifold points are not an essential part of the structure of $\boldsymbol{n}$**Cob**.

3. How To Do Category-Theoretic Physics

Ex. 2: The category **Hilb**.

- C-objects: finite-dim Hilbert spaces.
- Morphisms: bounded linear operators.

Set-theoretically:

- Hilbert spaces are structured sets.
- Structure-preserving functions are unitary operators.

Category-theoretically:

- C-objects of **Hilb** are not structured sets: general bounded linear operators need not preserve inner product.
- Unlike **Set**, **Hilb** admits a tensor product but no Cartesian product.
- C-elements of **Hilb** are vectors, but vectors are not an essential part of the structure of **Hilb**.

Objection: "...**Hilb** is certainly a concrete category, since the [C-]objects are Hilbert spaces, which are sets with extra conditions; and the morphisms are just functions with linearity conditions." (Lal & Teh 2017)

Response (Eva 2016)

- The relevant category-theoretic structure ("web of morphisms") encoded in **Hilb** is associated with the inner-product structure defined on its C-objects.
- This inner-product structure is not preserved by a functor that maps **Hilb** into **Set**.
- *So*: The fact that **Hilb** is, technically, a *concrete category* does not entail that it should be thought of as a category of structured sets.

3. How To Do Category-Theoretic Physics

Claim: $\boldsymbol{n}$**Cob** and **Hilb** are relevant to the pursuit of unifying descriptions of physical phenomena.

- A *topological quantum field theory* (TQFT) is a functor

$$Z : \boldsymbol{n}\mathbf{Cob} \rightarrow \mathbf{Hilb}$$

- To every $(n-1)$-dim manifold S, Z assigns a Hilbert space $Z(S)$.
- To every n-dim cobordism $M : S \rightarrow S'$, Z assigns a linear operator $Z(M) : Z(S) \rightarrow Z(S')$.
- $Z(M'M) = Z(M')Z(M)$, for any n-dim corbodisms M, M'.
- $Z(1_S) = 1_{Z(S)}$, for any $(n-1)$-dim manifold S.

4. Conclusion

Claim 1 (Mathematical): Category theory provides the means of mathematically representing structures independently of objects.

Claim 2 (Physical): *One* reason to believe that there are physical structures that are independent of objects is that there are theories in physics that

(a) warrant our belief,

(b) we should read literally, and

(c) employ category theory to represent structures independently of objects.

4. Conclusion

Provisos

- Defense of ROSR interpretation of theories in physics against charge of incoherence.

- Not an argument that necessitates ROSR interpretation.
  - Assumes metaphysical naturalism.
  - Assumes semantic realism.
  - Does not rule out alternative OSR intepretations
    - Balanced OSR. (Lam & Wuthrich 2015)

References

- Arenhart & Bueno (2015) 'Structural realism and the nature of structure', *EJPS 5*, 111.
- Baez (2006) 'Quantum quandries: A category-theoretic perspective', in D. Rickles *et al.* (eds.) *The structural foundations of quantum gravity*, Oxford, 240.
- Dorato (2016) 'The physical world as a blob: Is OSR really realism?', *Metasci 25*, 173.
- Dorato (2006) 'Is structural spacetime realism relationism in disguise?', in D. Dieks (ed.) *The ontology of spacetime*, Elsevier, 17.
- Esfeld (2015) Review of *The structure of the world*, *Mind 124*, 334.
- Esfeld & Lam (2008) 'Moderate structural realism about spacetime', *Synthese 160*, 27.
- Eva (2016) 'Category theory and physical structuralism', *EJPS 6*, 231.
- French (2014) *The structure of the world*, Oxford.
- French & Ladyman (2003) 'Remodeling structural realism', *Synthese 136*, 31.
- Heller & Sasin (1995) 'Structured spaces and their application to relativistic physics', *J Math Phys 36*, 3644.
- Lal & Teh (2017) 'Categorical generalization and physical structuralism', *BJPS* 68, 213.
- Lam & Wüthrich (2015) 'No categorical support for ROSR', *BJPS 66*, 605.
- Mac Lane (1998) *Categories for the working mathematician, 2nd Ed.*, Springer.
- Ney (2014) Review of *The structure of the world*, *Notre Dame Phil Rev.*
- Stachel (2006) 'Structure, individuality, and quanum gravity', in D. Rickles *et al.* (eds.) *The structural foundations of quantum gravity*, Oxford, 83.