Revenge on Hájek on Pascal's Revenge

Eddy Keming Chen

Department of Philosophy,
Rutgers University, New Brunswick

eddy.chen@rutgers.edu

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Introduction

My reading of Alan Hájek:

There is a newly discovered foundational problem in standard Expected Utility (EU) Theory: it is prone to collapse.

The collapse problem becomes vivid when we focus on some decision paradoxes:

1. Pascal's Wager
2. The St. Petersburg Game
3. The Pasadena Game
4. Paul's Wager

Introduction

Two types of collapse:

1. Decision problems involving infinite values or infinite state spaces. $\rightarrow$ same / undefined EU for every action.
2. Decision problems involving value gaps. $\rightarrow$ undefined EU for every action.

Either way, EU theory collapses, as it offers no discrimination between better and worse actions. In no situation can it offer you any practical advice about what you should do.

Introduction

In these comments, I hope to offer some constructive ideas about how to defend EU theory.

1. To avoid "collapse by infinity": maybe we can make do with some new math.
2. To avoid "collapse by value gaps": new math won't really help here; but we can make a different response.

Pascal's Wager

	God	No God	Expected Payoff
Wager for God	∞	f_1	Infinite
Wager against God	f_2	f_3	Finite

Table: Pascal's Wager

EU(mixed strategy) = P(wagering for God) $\times$ EU(wagering for God) + P(wagering against God) $\times$ EU(wagering against God)

$$= p\infty + (1-p)f_4 = \infty$$

Therefore, any mixed strategy of wagering for / against God also has infinite expected utility. All actions have the same EU value ∞. Any practical guidance?

Alan's Beer

Even having a beer might (non-zero probability) lead to playing Pascal's Wager :-(

Pascal's Wager

$p\infty + (1-p)f_4 = \infty$. That is, when we use real-number representation of credences and utilities.

The problems that Alan raised are serious. EU theory used to be a systematic way to rationalize our behavior. Now it's not.

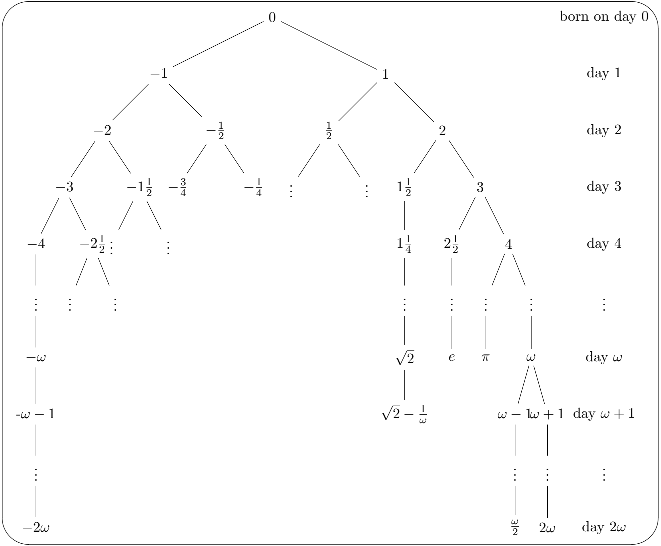
Proposal: save it with better tech (mathematical representation). In Chen and Rubio (ms.), we propose a surreal decision theory.

A Surreal Solution

Surreal Numbers (John Conway, 1976):

1. an ordered-field including all reals and ordinals (in the sense that their ordered fields can be realizes as subfields of the surreals);
2. addition in that field that is commutative, non-absorptive, and such that each element has an additive inverse;
3. multiplication in that field that is commutative, non-absorptive, and such that each non-zero element has a multiplicative inverse.

The Surreal Tree Structure



Question: Can we use surreal numbers in decision theory?

Answer: Yes, we prove a surreal von Neumann-Morgenstern Representation Theorem.

SURREAL VON NEUMANN-MORGENSTERN THEOREM: Let X be a space of lotteries, and let $\preceq$ be a binary relation $\subseteq X \times X$. There exists an affine function $U : X \to \mathbf{No}$ such that $\forall x, y \in X$

$U(x) \leq U(y) \Leftrightarrow x \preceq y$

if and only if $\preceq$ satisfies all of the following:

1. Completeness: $\forall x, y \in X$, either $x \preceq y$ or $y \preceq x$.
2. Transitivity: $\forall x, y, z \in X$, if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.
3. Continuity$\star$: $\forall x, y, z \in X$, if $x \prec y \prec z$, then there exist surreals $p, q \in \star(0, 1)$ such that $px + (1 - p)z \prec y \prec qx + (1 - q)z$.
4. Independence$\star$: $\forall x, y, z \in X, \forall p \in\star$(0, 1],x $\preceq y$ if and only if $px + (1 - p)z \preceq py + (1 - p)z$.

Proof: Chen and Rubio (ms.), Appendix.

Pascal's Wager

Real decision theory: $p\infty + (1 - p)f_4 = \infty$.

Surreal decision theory: $p\omega_1 + (1 - p)f_4 < \omega_1$.

- According to the surreal decision theory, not all mixed strategies have the same expected utility.
- They are sensitive to credences just as they should be.
- No collapse on the surreal decision theory.

The St. Petersburg Game

The collapse resulting from St. Petersburg Game can be similarly handled in the surreal decision theory.

The Pasadena Game

It might be natural to think that surreal representation solves all the problems. But that would be too naive.

The Pasadena Game presents new difficulties. As Alan points out, the problem (Riemann Rearrangement Theorem) comes from the order dependency of infinite sum of conditionally convergent series on $\mathbb{R}$, which also affects surreal infinite sum on **No**.

The Pasadena Game

Example: The Pasadena Game Payoff Series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... = \ln(2)$$

A rearrangement that gives us ∞:

$$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{6} + ...$$

$$> \frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{1}{6} + \frac{1}{4} - \frac{1}{6} + \frac{1}{4} - \frac{1}{6}.... = \infty$$

The Pasadena Game

The kind of rearrangement that changes the value of the infinite sum seems to share a common feature: intuitively speaking, they pack in "more" positive terms than negative terms, or vice versa.

Of course, standard analysis cannot distinguish the sense of "more" here, as there are as many positive terms as there are negative ones—both are infinitely many.

The Pasadena Game

Perhaps we can capture the sense of "more" using new tech. In probabilistic number theory, there exists a notion of natural density that seems to capture the intuition here.

$$nd(A) = \lim_{n\to\infty} \frac{|A \bigcap \{1, 2, ..., n\}|}{n}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... = \ln(2)$$

$O_1 = \{2, 4, 6, 8, ...\}$. $nd(O_1) = 1/2$.

$$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{6} + ... = \infty$$

$O_2 = \{2, 5, 10, ...\}$. $nd(O_2) < 1/2$.

The Pasadena Game

Conjecture: order dependency is removed when we compare infinite series with the same natural density. In other words: for any infinite series, no rearrangement will change its sum without changing its natural density.

If our conjecture is true, then we can restore definiteness into surreal decision theory, without making every rearrangement a different game.

1. Collapse by Infinities
2. Collapse by Value Gaps

Paul's Wager

	God	No God	Expected Payoff
Convert to Religion	?	?	?
Stay Non-Religious	f_1	f_2	Finite

Table: Paul's Wager

The EU is ?. Any mixed strategy has EU ?. Any action is a mixed strategy of Paul's Wager. Thus, EU theory collapses.

Moreover, anyone with non-zero credence in Paul's conclusion (TE are EU-gaps) gets the same result. $\rightarrow$ Paul's revenge.

Paul's Wager

Initial worry: Paul's revenge seems to backfire. Paul wants to maintain that deliberating about transformative experiences are *sui generis* — in a unique category of deliberation.

But it looks like everything has become a mixed strategy of deliberating about transformative experiences. So, those TE decisions are not unique anymore.

Paul's Wager

More serious worry: decision theory, surrealized or not, seems unable to rescue itself from the collapse. The problem is not about mathematical representation.

What about assigning probability 0 to Paul's conclusion (TE are EU-gaps)?

Alan: "it is not even clear that you can avoid [collapse] by assigning probability 0 – but in any case, are you rationally entitled to be so dogmatic?"

Paul's Wager

Suppose we can avoid collapse by assigning probability 0 to Paul's conclusion (TE are EU-gaps).

I think it is rationally permissible for us to do so.

1. Suppose it is a rational requirement that cr(Paul's conclusion) > 0.
2. Then, necessarily, it is a rational requirement that cr(Paul's conclusion) > 0.
3. Theorem: If it is a rational requirement that cr(Paul's conclusion) > 0, then all actions have undefined EU.
4. Corollary 1: Necessarily, all actions have undefined EU.
5. Corollary 2: EUT is impossible.
6. Thus, it is a rational requirement that cr(EUT is possible) $= 0$.

Are we rationally required or even permitted to be so dogmatic, committing ourselves to the impossibility of EUT, when EUT

Paul's Wager

So, it looks like we are choosing between two dogmas:

1. cr(Paul's conclusion) $= 0$.
2. cr(EUT is possible) $= 0$.

In this choice situation, it seems to me that we are rationally permitted to choose (1) over (2).

If $0\times\ ?\ =\ ?$, then we can run a similar argument. We can argue that we are rationally permitted to treat Paul's conclusion as impossible and remove it from the σ-algebra.

Note: In the first three examples of EUT collapse, I don't think we are rationally permitted for this response, because there seem to be better alternatives available.

Alan's Beer

So: there is still hope that we can have a beer without crashing EU theory :-)