On Mathematical Structuralism

A Theory of Unlabeled Graphs as *Ante Rem* Structures

Hannes Leitgeb

LMU Munich

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Mathematics as a science of structures:

> *If in the consideration of a simply infinite system N set in order by a transformation φ we entirely neglect the special character of the elements, simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation, then are these elements called natural numbers... (Dedekind 1888)*

Dedekind on arithmetic, analysis
Hilbert on geometry
Zermelo on set theory
} Structures, (quasi-)categorical axiom systems

Logic
Bourbaki on algebra, topology,...
Category theory
} General "structures"

Structuralism in the philosophy of mathematics:

> *[I]n giving the properties... of numbers you merely characterize an abstract structure... [T]he "elements" of the structure have no properties other than those relating them to other "elements" of the same structure... To be the number 3 is no more and no less than to be preceded by 2, 1, and possibly 0, to be followed by 4, 5, and so forth... (Benacerraf 1965)*

Goal:

- Make one version of structuralism—*ante rem* structuralism—precise by means of an example theory, and argue that it is coherent.

Plan:

1. A Theory of Unlabeled Graphs as *Ante Rem* Structures
2. Philosophical Assessment of the Theory
3. Unlabeled Graph Theory vs Set Theory
4. Conclusions

We will concentrate on ontological and semantic questions about structures: *what are structures*, and *how can we speak about them*?

A Theory of Unlabeled Graphs as *Ante Rem* Structures

Unlabeled graph: a graph in which individual nodes have no distinct identifications except through their interconnectivity. (Wolfram MathWorld)

two graphs $G = (N, E)$ *and* $H = (N, F)$ *are the same unlabeled graph when they are isomorphic... (Mahadev & Peled 1995, Threshold Graphs and Related Topics)*

Sometimes we are interested only in the "structure" or "form" of a graph and not in the names (labels) of the vertices and edges. In this case we are interested in what is called an unlabeled graph. A picture of an unlabeled graph can be obtained from a picture of a graph by erasing all of the names on the vertices and edges. This concept is simple enough, but is difficult to use mathematically because the idea of a picture is not very precise. (Bender & Williamson 2010, Lists, Decisions and Graphs)

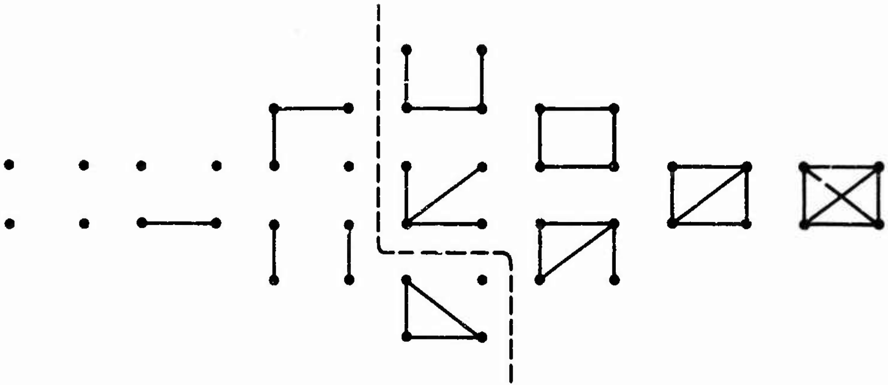


Fig. 2.1. The graphs with four points.

(Harary 1969, *Graph Theory*)

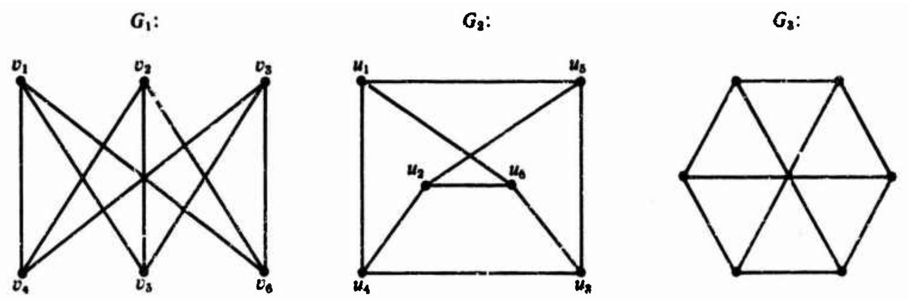


Fig. 2.5. Labeled and unlabeled graphs.

A graph G is labeled when the p points are distinguished from another by names such as v_1, v_2,..., v_p, For example, the two graphs G_1 and G_2 of Fig. 2.5 are labeled but G_3 is not.

Rather than continue with an intuitive development of additional concepts, we proceed with the tedious but essential sequence of definitions upon definition. (Harary 1969)

We seem to have a clear intuition ("*Anschauung*") of unlabeled graphs.

But that intuition is not preserved by the set-theoretic definition of 'graph':

- A graph is a pair $\langle V, E\rangle$, such that $V \neq \emptyset,\ E \subseteq \{\{v, w\} | v, w \in V, v \neq w\}$.

  E.g.:

  $\langle\{0,1,2\},\{\{0,1\},\{1,2\}\}\rangle$ (= $\{\{\{0,1,2\}\},\{\{0,1,2\},\{\{0,1\},\{1,2\}\}\}\}$).
  $\langle\{\pi,e,0\},\{\{\pi,e\},\{e,0\}\}\rangle$ (= $\{\{\{\pi,e,0\}\},\{\{\pi,e,0\},\{\{\pi,e\},\{e,0\}\}\}\}$).

- A labeled graph is a triple $\langle V, E, l\rangle$, such that $\langle V, E\rangle$ is a graph and l: $V \to \mathbb{N}$.

In the following, we are going to state an axiomatic theory UGT of unlabeled graphs (undirected, without loops or multiple edges) as structures *sui generis*.

Language:

- language of second-order logic with identity.

  Primitive predicates: '$Graph(G)$', '$Vertex(v, G)$', '$Connected(v, w, G)$'.

  'G', 'v': first-order variables (sometimes we use 'G' restricted to graphs).

  'X', 'R', 'f': second-order variables.

Intended first-order universe D:

- unlabeled graphs and their vertices.

Intended second-order universe:

- sets, relations, functions on D.

Logic: standard deductive system of second-order logic; in particular:

- $\exists X \forall x(X(x) \leftrightarrow \varphi[x])$ (X not free in φ).

  E.g.: $\forall G \exists X \forall x(X(x) \leftrightarrow \mathit{Vertex}(x, G))$.

  φ is functional $\rightarrow \exists f \forall v, w(f(v) = w \leftrightarrow \varphi[v, w])$ (f not free in φ)

- $\forall x, y$: $x = y \leftrightarrow \forall X(X(x) \leftrightarrow X(y))$.

  $\forall X, Y$: $X = Y \leftrightarrow \forall x(X(x) \leftrightarrow Y(x))$.

  E.g.: $\forall G \exists ! X \forall x(X(x) \leftrightarrow \mathit{Vertex}(x, G))$.

  $\forall f, g$: $f = g \leftrightarrow \forall x(f(x) = g(x))$.

- Choice Axiom:

  $\forall R^{n+1}(\forall x_1, \ldots, x_n \exists y R^{n+1}(x_1, \ldots, x_n, y) \rightarrow$
  $\exists f^n \forall x_1, \ldots, x_n R^{n+1}(x_1, \ldots, x_n, f(x_1, \ldots, x_n)))$.

Various definitions, e.g.:

- $\forall v, \forall G$: $\mathit{Isolated}(v, G) \leftrightarrow \mathit{Vertex}(v, G) \wedge \neg\exists w\, \mathit{Connected}(v, w, G)$.
- $\forall G, \forall X$: $V(G) = X \leftrightarrow \forall x(X(x) \leftrightarrow \mathit{Vertex}(x, G))$.

General axioms for unlabeled graphs:

- $\forall G \forall v, w$: $Connected(v, w, G) \rightarrow$

  (i) $Vertex(v, G) \wedge Vertex(w, G)$,

  (ii) $v \neq w$,

  (iii) $Connected(w, v, G)$.

- $\forall G \forall v$: $Vertex(v, G) \rightarrow$

  $\neg \exists G' (G' \neq G \wedge Vertex(v, G')) \wedge \neg Graph(v)$.

- Identity criterion: $\forall G, G'$: $G = G' \leftrightarrow G \cong G'$.

  With:

  - $G \cong G' \leftrightarrow \exists f(f\colon G \rightarrow G' \wedge f \text{ bijective}_{G,G'} \wedge f \text{ structure-preserving}_{G,G'})$.
  - $f\colon G \rightarrow G' \leftrightarrow \forall v(Vertex(v, G) \rightarrow Vertex(f(v), G'))$.
  - $f \text{ bijective}_{G,G'} \leftrightarrow \forall w(Vertex(w, G') \rightarrow \exists! v(Vertex(v, G) \wedge f(v) = w))$.
  - $f \text{ structure-preserving}_{G,G'} \leftrightarrow \forall v, w(Vertex(v, G) \wedge Vertex(w, G) \rightarrow (Connected(v, w, G) \leftrightarrow Connected(f(v), f(w), G')))$.

Existence axioms for unlabeled graphs:

- $\exists G \exists ! v Vertex(v, G)$.

  The existence of the trivial graph (Harary, p.9).

- $\forall G \exists G' \exists v'$, such that:
  $Vertex(v', G')$,
  $Isolated(v', G')$,
  $\exists f\ Isomorphism(f, G, G' - v')$.

  $G' - v'$: removal of a point (Harary, p.11).

- $\forall G \forall v, w\, (Vertex(v, G) \wedge Vertex(w, G) \wedge v \neq w \wedge \neg Connected(v, w, G) \rightarrow$
  $\exists G' \exists v' \exists w'$, such that:
  $Connected(v', w', G')$,
  $\exists f (Isomorphism(f, G, G' - \{v', w'\}) \wedge v' = f(v) \wedge w' = f(w))$.

  $G' - \{v', w'\}$: removal of a line (Harary, p.11).

On this basis we can prove all finite unlabeled graphs to exist, and we can determine the cardinality of unlabeled graphs with a fixed number of vertices.

Theorem

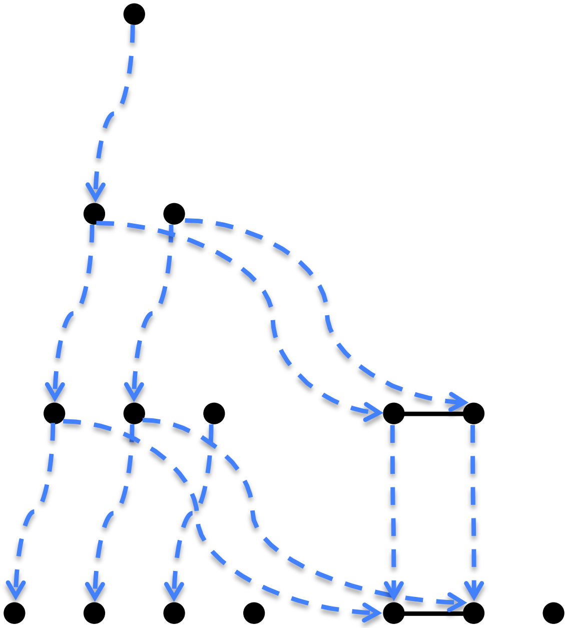
E.g., using UGT, we can derive:

- $\exists! G_0 \exists! v\, Vertex(v, G_0)$.
- $\exists! G_1 \exists v_1, v_2 (v_1 \neq v_2 \wedge Vertex(v_1, G_1) \wedge Vertex(v_2, G_1) \wedge \neg Connected(v_1, v_2, G_1) \wedge \forall w (Vertex(w, G_1) \rightarrow w = v_1 \vee w = v_2))$.
- *There exist precisely four unlabeled graphs with three vertices.*

Proof: Combination of existence axioms and identity criterion.

(All of these finite unlabeled graphs can be described categorically, of course.)

Accordingly, one can determine the right number of automorphisms for a given unlabeled graph; and so on.



Metatheorem

UGT is consistent.

Proof: Provide set-theoretic model.

- First-order domain D: For each isomorphism type of finite set-theoretic graphs with vertices in $\mathbb{N}$, pick one member; but do so in a way such that no two picked set-theoretic graphs share a vertex.

  Put these set-theoretic graphs into D as well as their vertices.

- Second-order domain: all sets, relations, functions on D.

- Interpret '$Graph(G)$', '$Vertex(v, G)$', '$Connected(v, w, G)$' as expected.

Extensions of the system:

- Introduce natural numbers and functions from vertices to natural numbers: state the second-order Dedekind-Peano axioms; include the natural numbers in the intended universe.

  Based on this, we can define, e.g.:

  f is a walk in G iff $\exists x(Nat(x) \wedge \forall y(Nat(y) \wedge y \leq x \rightarrow Vertex(f(y), G)) \wedge$
  $\forall y(\neg Nat(y) \vee y > x \rightarrow y = G_0) \wedge$
  $\forall y(Nat(y) \wedge y < x \rightarrow Connected(f(y), f(y+1), G)))$.

  Define: connectedness, length of walk, distance, degree, etc.

  One can define recursive functions on graphs explicitly, prove theorems by induction (e.g. over the number of vertices of graphs), and derive in this way theorems about all finite unlabeled graphs.

Relations between graphs:

- G' is a subgraph of G if and only if $\exists X \exists f (\forall v (X(v) \rightarrow Vertex(v, G)) \wedge Isomorphism(f, G|_X, G'))$.

Further graph-theoretic operations:

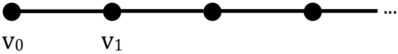
- Subgraph axiom:

$$\forall G \forall X (\forall v (X(v) \rightarrow Vertex(v, G)) \rightarrow \exists G' \exists f\, Isomorphism(f, G|_X, G')).$$

$Isomorphism(f, G|_X, G')$:

$$\begin{aligned}
&\forall x (X(x) \rightarrow Vertex(f(x), G')) \wedge \\
&\forall x' (Vertex(x', G') \rightarrow \exists! x (X(x) \wedge f(x) = x')) \wedge \\
&\forall x, y (X(x) \wedge X(y) \rightarrow (Connected(x, y, G) \leftrightarrow \\
&\qquad Connected(f(x), f(y), G'))).
\end{aligned}$$

- Union graph axiom (not literal union!)
- Product graph axiom (use category-theoretic formulation!)
- ⋮



- Infinity graph axiom:

$$\exists G \exists v_0, v_1 \colon \mathit{Vertex}(v_0, G) \wedge \mathit{Vertex}(v_1, G) \wedge \mathit{Connected}(v_0, v_1, G) \wedge$$
$$\forall w(\mathit{Connected}(w, v_0, G) \rightarrow w = v_1) \wedge$$
$$\exists f(\mathit{Isomorphism}(f, G, G - v_0) \wedge f(v_0) = v_1).$$

($\rightarrow$ There exists a special graph which is *identical* to one of its subgraphs.)

Use subgraph axiom to determine the (structurally) least graph of that kind.

Philosophical Assessment of the Theory

This theory of unlabeled graphs should count as an axiomatic treatment of (a special family of) *ante rem* structures:

> *If in the consideration of a simply infinite system N set in order by a transformation φ we entirely neglect the special character of the elements, simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation, then are these elements called natural numbers... (Dedekind 1888)*

Let us now discuss some of the standard worries about *ante rem* structuralism against the background of our theory:

The problems of *identity*, *objects*, *axiomatization*, *reference*.

- If two objects a, b in a structure are structurally indistinguishable—that is, there is an automorphism f so that $f(a) = b$—shouldn't they be identical?

  Which would be against mathematical practice (e.g., i vs $-i$).
  cf. Burgess (1999), Keränen (2001).

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  $\hookrightarrow$ *No!* (cf. Ketland 2006, Leitgeb & Ladyman 2008, Shapiro 2008.)

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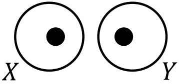
Metaphysically: Let a, b so that $Vertex(a, G_1)$, $Vertex(b, G_1)$, and $a \neq b$.

The fact that $a \neq b$ obtains in virtue of

$$\exists v_1, v_2(v_1 \neq v_2 \wedge Vertex(v_1, G_1) \wedge Vertex(v_2, G_1) \wedge \neg Connected(v_1, v_2, G_1) \wedge \forall w(Vertex(w, G_1) \rightarrow w = v_1 \vee w = v_2)).$$

That $a \neq b$ holds is grounded in what the unlabeled graph G_1 is like. ✓



On the logical side, our logical identity principles are perfectly consistent with our *ante rem* structuralism about unlabeled graphs:

– $\forall x, y\colon x = y \leftrightarrow \forall X(X(x) \leftrightarrow X(y))$. (PII)

Don't restrict PII to "qualitative" properties, don't think of it predicatively. (*Identity* is perfectly structural, just as *number of vertices in graph G* is!)

– $\forall X, Y\colon X = Y \leftrightarrow \forall x(X(x) \leftrightarrow Y(x))$. (Extensionality)

Don't misunderstand this to be about the identity of *subgraphs*!
(Within a G, 'X' and 'Y' range over *sets* of vertices or, plurally, over *vertices*.)

- “Objects” (“places”) in *ante rem* structures are not really *objects*.

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  $\hookrightarrow$ Vertices in unlabeled graphs may not be substances/individuals in a traditional metaphysical sense (cf. Caulton and Butterfield 2012),
  but they are objects in a logical or Quinean sense:

  – they are (members of) values of bound variables ($\exists v$, $\forall v$, $\exists X$, $\forall X$),

  – one can map them to other objects,

  – there is an identity/difference relation for them,

  – one can count them. $\checkmark$

- There are no precise axioms for *ante rem* structures, or such axioms are just “set theory in disguise” (such as the axioms in Shapiro 1997).

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  $\hookrightarrow$ Done (for unlabeled graphs). $\checkmark$

  The axiomatic system UGT for unlabeled graphs is

  (i) in line with pre-set-theoretic mathematical practice,
  (ii) clear, exact, systematic,
  (iii) based on a structuralist identity criterion,
  (iv) formulated structurally ("along structure-preserving maps"),
  (v) consistent,
  (vi) can easily be strengthened.

  (What else would we have to deliver?)

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  Is it also possible to refer to all of the *nodes* in non-rigid graphs (where there exists an automorphism f, such that $f(a) = b$ and $a \neq b$)?

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  cf. Shapiro (2008), Räz (2013). $\hookrightarrow$ Yes! ✓ (cf. Brandom 1996)

Informally: Let 'a' denote any node in G_1; let 'b' denote the other node.

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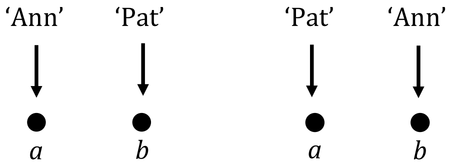
  Formally: $a = \varepsilon v Vertex(v, G_1)$; $b = \varepsilon v(Vertex(v, G_1) \wedge v \neq a)$.

  cf. Hilbert, Bourbaki, Carnap, Shapiro (2012) on epsilon terms.

  Given the ε-term definitions of 'a' and 'b', our previous 'the fact that $a \neq b$ obtains in virtue of...' can be made precise in terms of 'is derivable from'.

Semantically:



Even Shapiro (2008) got this wrong when he said "There simply is no naming any point... in some graphs".

There are simply *distinct* but *structurally indistinguishable* reference relations on G_1, just as a and b are themselves distinct and structurally indistinguishable.

(Or maybe this is not "real" reference? Well,...)

Unlabeled Graph Theory vs Set Theory

How does UGT relate to set theory?

Set theory has three roles to play in modern mathematics:

1. Set theory as a mathematical “language”:

   e.g., second-order quantifiers $\forall X$, $\exists X, \forall f, \exists f$ in UGT. ✓

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3. Set theory as a foundation of mathematics:

   Interpreted axiomatic set theory plus methods of reducing mathematical objects, concepts, theorems to set-theoretic objects, concepts, theorems.

   How this squares with UGT depends on the *purpose* of reduction.

Set theory as a foundation of mathematics:

3a For *mathematical purposes:* show that reduction is possible and how so.

Relative interpretability yields (i) relative consistency, (ii) ways of using methods and results from one field in another.

E.g., proof of consistency of UGT ✓

But note that, with a bit more work, the direction of reduction may also be reversed: one might show that set theory is reducible to UGT!

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3b For *quasi-philosophical purposes:* determine purely set-theoretically the (uniquely determined) *intended* interpretation of mathematical language.

This I find questionable! $\not\checkmark$
Relative interpretation preserves derivability but not necessarily meaning.

just *simulated* by $\{\{\{0,1,2\}\},\{\{0,1,2\},\{\{0,1\},\{1,2\}\}\}\}$
(or its isomorphism class)

Conclusions

- Unlabeled graphs can be treated, mathematically and philosophically, as *structures sui generis*.
- At least as far as unlabeled graphs are concerned, *ante rem* structuralism amounts to a *coherent* position.
- None of this is against set theory *per se*: just against taking the set-theoretic reconstruction of the *basic structures* too seriously.
- From the point of view of this theory, *ante rem* structuralism about mathematics is *not* about:

  objects vs higher-order entities;

  properties vs relations;

  "the right" language of mathematics.