Indeterminacy at Large-Order

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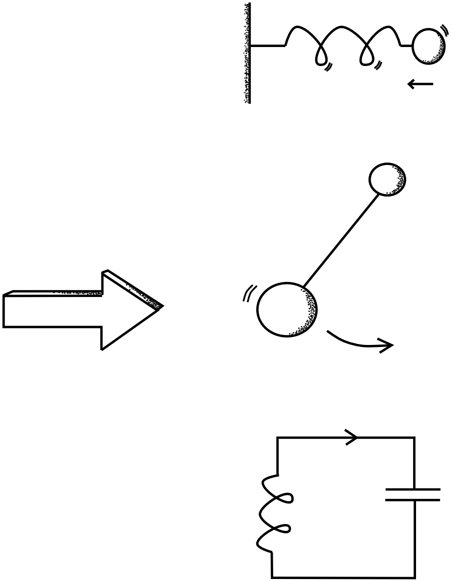
Physics has trained us to expect reality to be maximally precise.

Dynamical equation:

$$a\frac{d^2q(t)}{dt^2} = -bq(t)$$



Solution:

$$q(t) = A\cos(\omega t + \phi)$$

At some length scale, questions like “what is the position of the center-of-mass of the bob?” stop making sense.

The theory is more precise than it needs to be to get questions about the spring, the pendulum, and the circuit, correct.

If we move to a truly fundamental theory, then maximal precision will become necessary to get the world right.

Or so you might think.

> Any formal power series being asymptotic to infinitely many smooth functions, perturbative field theory alone does not give any well defined mathematical recipe to compute to arbitrary accuracy any physical number, so in a deep sense it is no theory at all.

(Magnen and Rivasseau 2008, p. 403)

Empirically adequate quantum field theories (our most fundamental theories) do not provide maximal precision.

Question: what should we make of this?

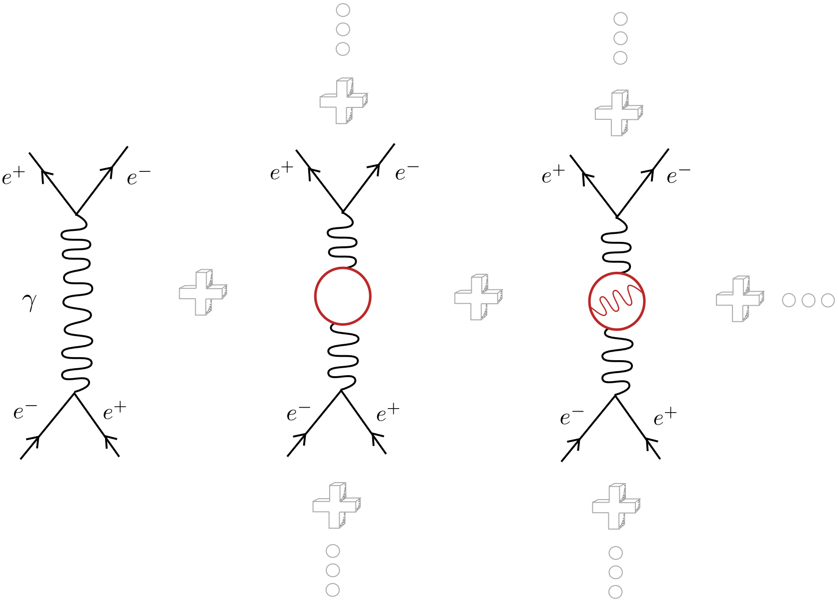
Part One: Indeterminacy at Large-Order

Part Two: Three Reactions to the Indeterminacy numbers

powers of the coupling

$$\text{s-matrix} \rightarrow \sum_{n=0}^{\infty} a_n \alpha^n$$

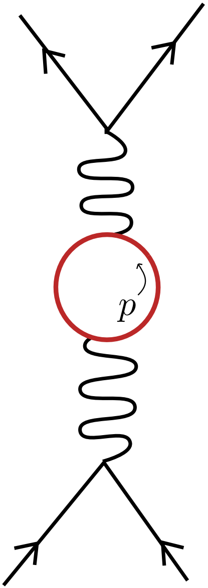
There are three sources of infinities that need to be addressed.



First order

Second order

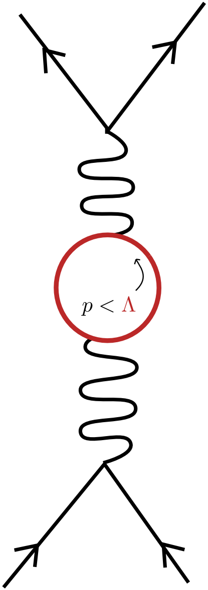
Third order



$$\int_0^\infty \frac{dp}{p} = \ln(\infty) = \infty$$

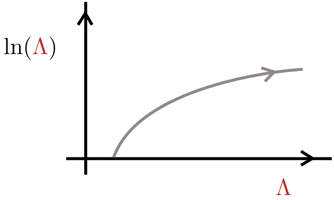
The theory predicts infinite probability?!?

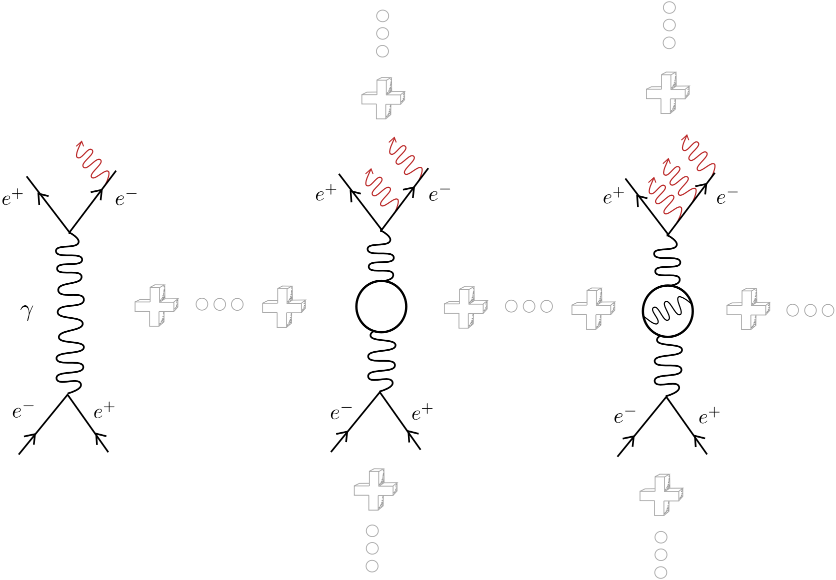
$$O_{\text{th}} \notin O_{\text{ex}} \pm \epsilon_{\text{ex}}$$



$$\int_0^{\Lambda} \frac{dp}{p} = \ln(\Lambda) = \text{finite}$$

(for finite Λ)

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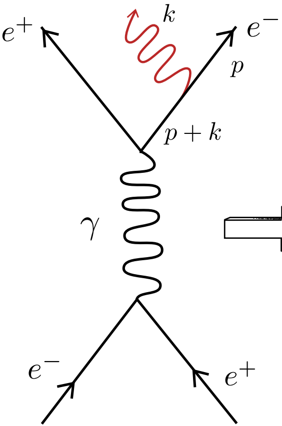
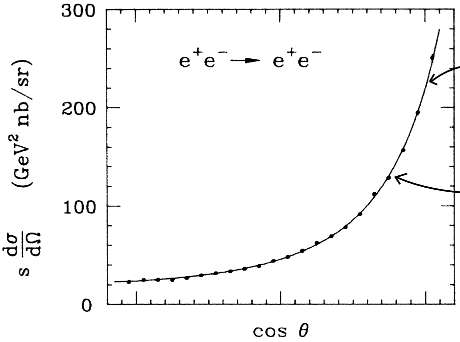


$$\int_0^\Lambda \frac{dk}{\sqrt{k^2+m_\gamma^2}} \sim \ln(\tfrac{\Lambda}{m_\gamma}) = \infty$$

for any finite Λ since $m_\gamma = 0$

$O_{\text{th}} \notin O_{\text{ex}} \pm \epsilon_{\text{ex}}$

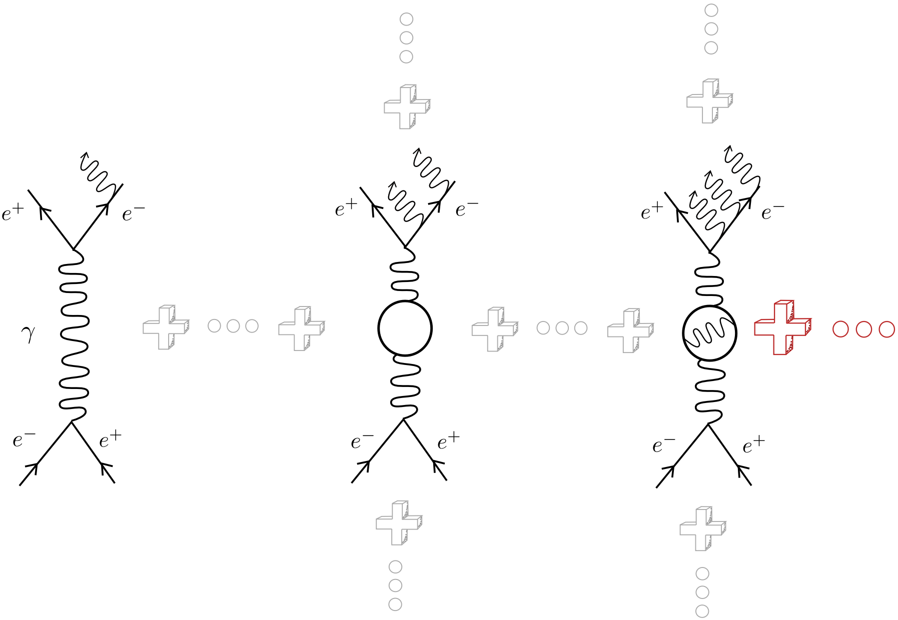


Sum of first three terms

Empirical data

$$O_{\mathrm{th}} \in O_{\mathrm{ex}} \pm \epsilon_{\mathrm{ex}}$$

M. Derrick et. al., Phys. Rev. **D34**, 3286 (1986).



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Perturbation theory converges if the sequence of partial sums $S_N = \sum_{n=0}^{N} a_n \alpha^n$ converges to a limit.

Obstacle: Quantum electrodynamics is large-order divergent.

F.J. Dyson, Phys. Rev. **85**, 631 (1952).

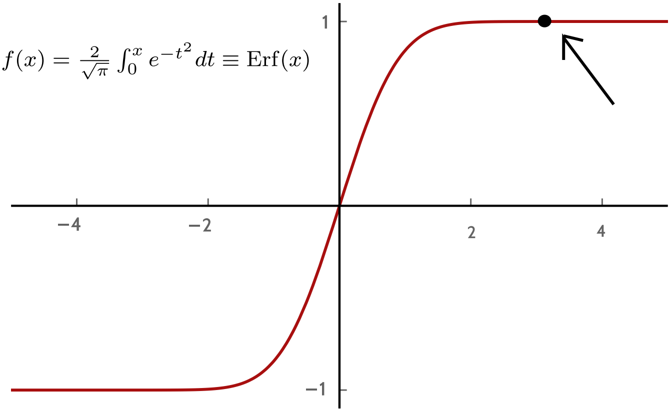
But if we stop at just a few orders of perturbation theory we get an answer that matches the empirical data very well.

What could be going on?

An explanation of this success is that perturbation theory is asymptotic to some exact structure.

Asymptoticity is a requirement that $f(x) - \sum_{n=0}^{N} a_n x^n$ is appropriately small at every order of perturbation theory.

$\lim_{x \to 0} \frac{f(x) - \sum_{n=0}^{N} a_n x^n}{x^N} = 0, \quad \forall N$

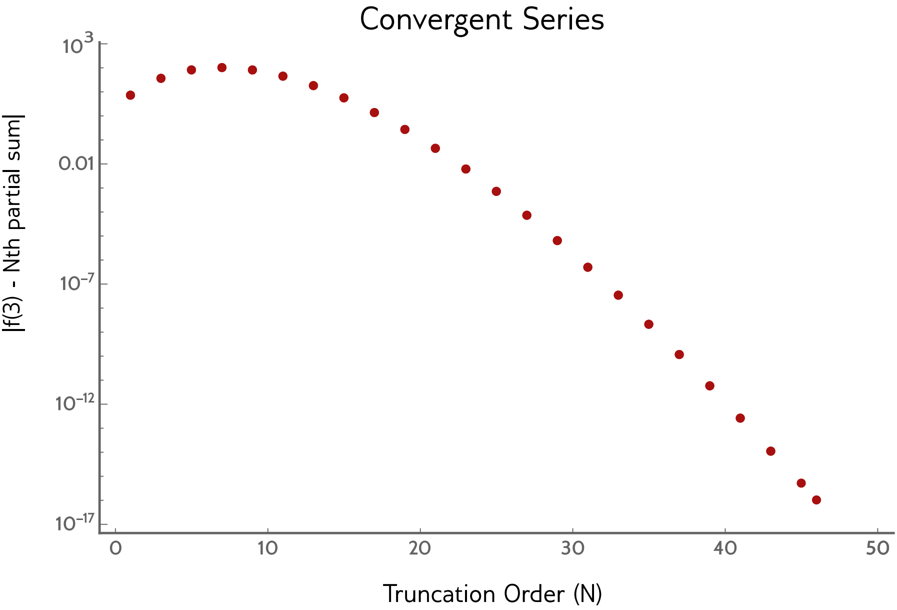


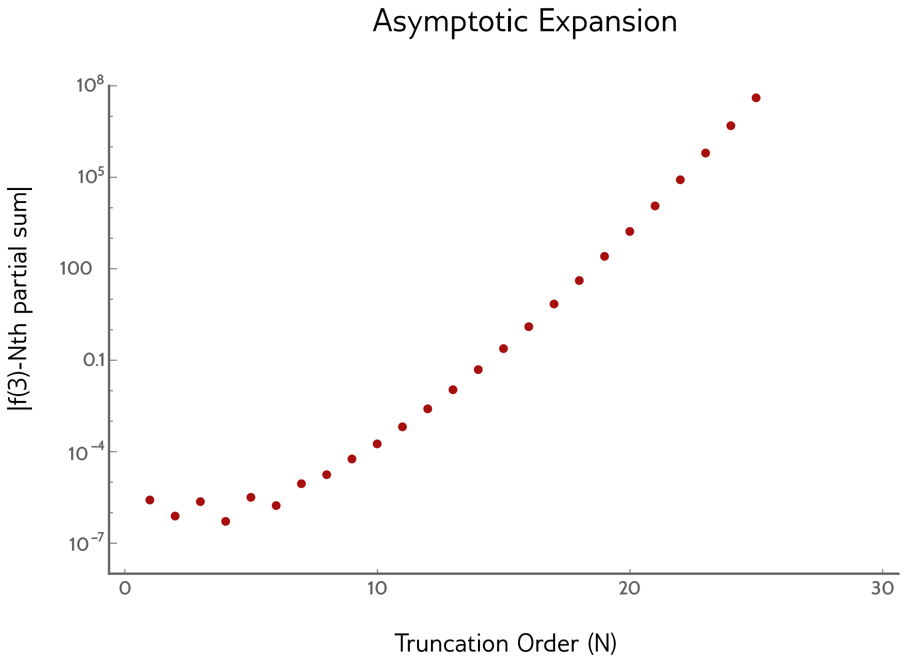
Convergent series:

$$f(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} x^{2n+1}$$

Asymptotic Series:

$$f(x) \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(2n-1)!!}{2^n} \frac{1}{x^{n+1}}$$


Convergent Series
10^3
0.01
10^{-7}
10^{-12}
10^{-17}
0
10
20
30
40
50
|f(3) - Nth partial sum|
Truncation Order (N)



Asymptotic Expansion
10^8
10^5
100
0.1
10^{-4}
10^{-7}
|f(3)-Nth partial sum|
0
10
20
30
Truncation Order (N)

How well does asymptoticity constrain exact structure?

$$\sum_{n=0}^{\infty} a_n \alpha^n \xrightarrow{?} \quad f_1(x) \quad f_2(x) \quad f_3(x) \quad \ldots$$

Many different functions have the same asymptotic expansion.

$$\sum_{n=0}^{\infty} a_n \alpha^n \longrightarrow \begin{matrix} & f_1(x) & \\ f_2(x) & & f_3(x) \\ & \cdots & \end{matrix}$$

Consider $h(x) = e^{-1/x}$ with $x > 0$.

Then $\frac{h(x)}{x^N} \to 0$ as $x \downarrow 0$.

So $h(x) \sim \sum_{n=0}^{\infty} 0 \cdot x^n = 0$

There is a condition stronger than asymptoticity which determines the function uniquely.

Strong asymptoticity requires that $f(z) - \sum_{n=0}^{N} a_n z^n$ is even smaller than required by asymptoticity.

$\exists C, \sigma$ such that $\left| f(z) - \sum_{n=0}^{N} a_n z^n \right| \le C \sigma^{N+1} (N+1)! |z|^{N+1} \;\; \forall N.$

A strong asymptotic series uniquely determines a function.

$$\sum_{n=0}^{\infty} a_n \alpha^n \longrightarrow \left\{ f_1(x),\ \cancel{f_2(x)},\ \cancel{f_3(x)},\ \ldots \right\}$$

If one knows a strong asymptotic series, the function can be uniquely reconstructed from the series by Borel summation.

Borel transform: divide the coefficients by $n!$.

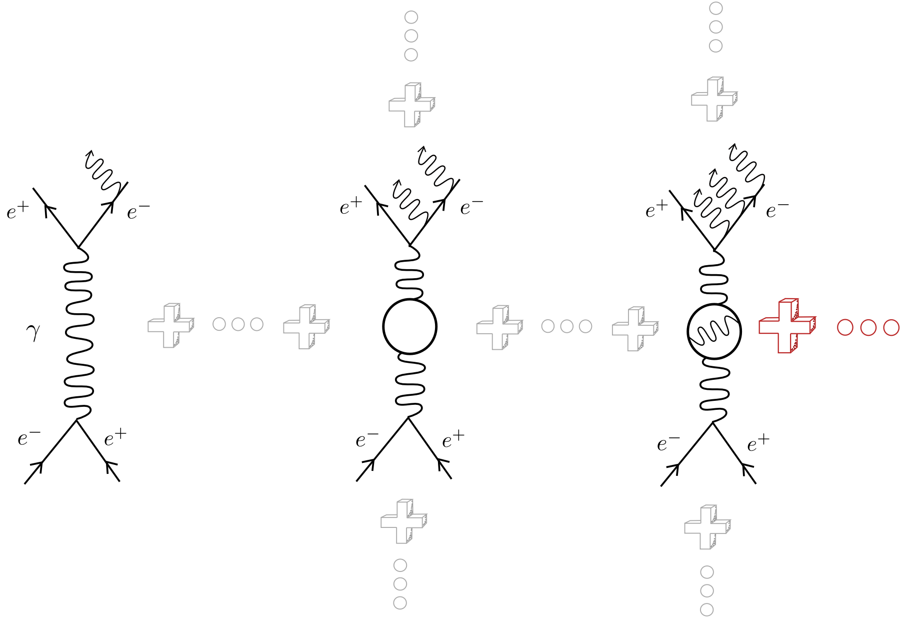
Borel summation: integrate to recover the exact function.

(Transform) $g(z') = \sum_{n=0}^{\infty} \frac{a_n}{n!} z'^n$

(Summation) $f(z) = \int_0^{\infty} g(xz) e^{-x} dx$

In the cases where a constructive model is available, perturbation theory exactly determines the model through Borel summation.

Can we define the Borel sum of empirically adequate models?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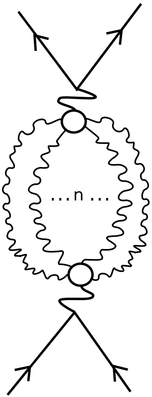
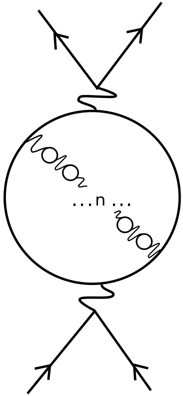
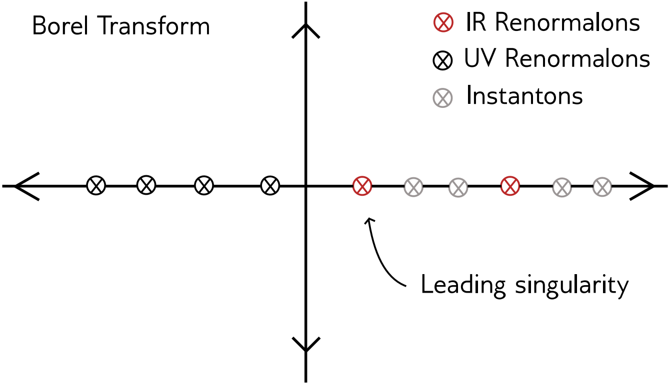




Instantons: a collection of $n!$ graphs each of which makes a small contribution

Renormalons: individual graphs that make a contribution like $n!$



G. 't Hooft. Erice Lecture (1977)

The division by $n!$ in the Borel transform is insufficient to completely tame the large-order divergent behavior.

There is no unique exact solution lying behind the empirical success of the truncated expansion.

$$\sum_{n=0}^{\infty} a_n \alpha^n \longrightarrow \quad f_1(x) \quad f_2(x) \quad f_3(x) \quad \dots$$

By optimally truncating the series, we can still obtain a value, but it is not maximally precise.

Part One: Indeterminacy at Large-Order

Part Two: Three Reactions to the Indeterminacy

Reaction One: Retreat to empirical adequacy.

Example: electron magnetic moment, a precision test of QED.

According to the Dirac equation, $g = 2$.

1947: measurements begin to show a small non-zero value of:

$$a_e = \frac{g-2}{2}$$

The Dirac equation fails to be emprically adequate.

The first-order perturbative correction from renormalized QED gives a non-zero value for a_e.

$$a_e \ (\text{theory}) = 0.001161$$

$$a_e \ (\text{experiment}) = 0.00115(4)$$

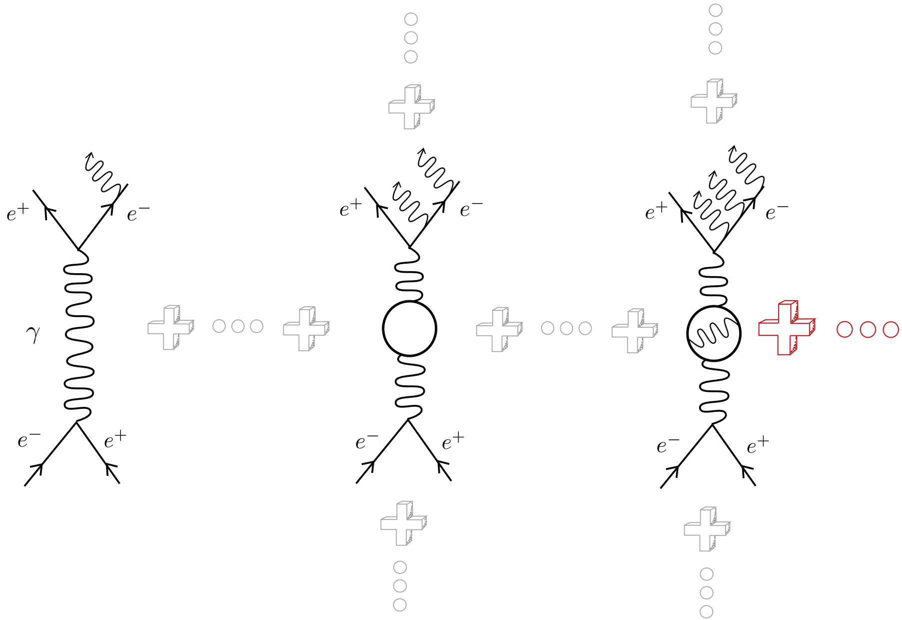
$$O_{\text{th}} \in O_{\text{ex}} \pm \epsilon_{\text{ex}}$$

(Theory) J. Schwinger, Phys. Rev. **72** (1947).
(Experiment) P. Kusch & H. M. Foley, Phys. Rev. Lett. **72** (1947).

$$a_e \text{ (theory)} = 0.00115965218178(77)$$

$$a_e \text{ (experiment)} = 0.00115965218073(28)$$

(Theory) T. Aoyama et. al. Prog. Th. Ex. Phys. **A01** 107 (2012).
(Experiment) D. Hanneke et. al. Phys. Rev. Lett. **100** 120801 (2008).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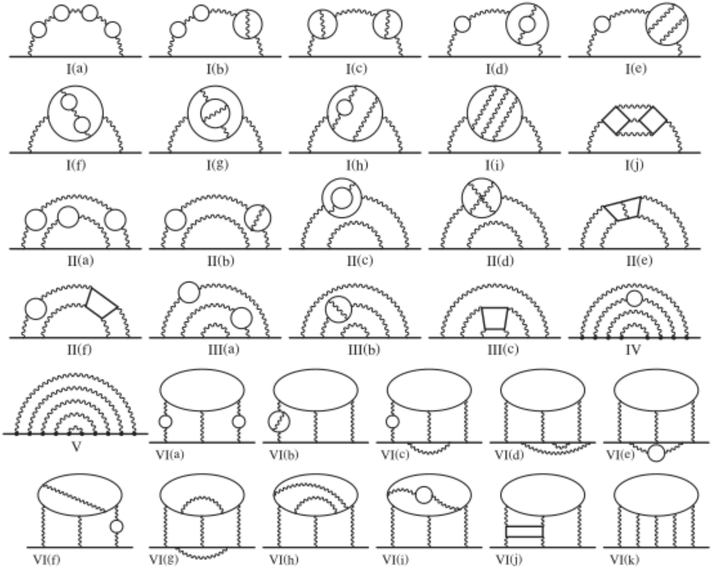
1st order $\rightarrow$ 1 diagram (treated analytically)

2nd order $\rightarrow$ 7 diagrams (treated analytically)

3rd order $\rightarrow$ 72 diagrams (treated analytically)

4th order $\rightarrow$ 891 diagrams (most treated numerically)

5th order $\rightarrow$ 12,672 diagrams (most treated numerically)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III(c)
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VI(b)
VI(c)
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VI(e)
VI(f)
VI(g)
VI(h)
VI(i)
VI(j)
VI(k)

a_e (theory) $= 0.00115965218178(06)(04)(02)(77)[77]$

(06) $\rightarrow$ numerical error from 4th order

(04) $\rightarrow$ numerical error from 5th order

(02) $\rightarrow$ hadronic and electroweak corrections

(77) $\rightarrow$ measured value of the fine-structure constant

Take empirical adequacy to be established when:

$$O_{\mathsf{th}} \pm \epsilon_{\mathsf{th}} \subset O_{\mathsf{ex}} \pm \epsilon_{\mathsf{ex}}$$

> Any formal power series being asymptotic to infinitely many smooth functions, perturbative field theory alone does not give any well defined mathematical recipe to compute to arbitrary accuracy any physical number, so in a deep sense it is no theory at all.

(Magnen and Rivasseau 2008, p. 403)

We have recovered a sense in which perturbative field theory is in fact a theory. But can we say more?

Reaction Two: There is a unique correct exact theory, it is just radically underdetermined by the low-order perturbative data.

On this view, the perturbative expansion is an approximation to some exact non-perturbative theory.

There is no unique exact solution lying behind the empirical success of the truncated expansion.

$$\sum_{n=0}^{\infty} a_n \alpha^n \longrightarrow \begin{array}{c} f_1(x) \\ f_2(x) \quad f_3(x) \\ \cdots \end{array}$$

Moreover, we don't have our hands on any of the fs. The idea that what we have is an approximation is simply a conjecture.

Reaction Three: The low-order perturbative data includes everything there is to know about the world.

If, counterfactually, perturbative field theory turned out to be the fundamental truth about the world, then we would have had genuine instances of metaphysical indeterminacy.

Physics has trained us to expect reality to be maximally precise. But should we trust our training?